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Sobolev Orthogonal Polynomials with a Small Number of Real Zeros

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Let $\{S_n^\lambda\}$ denote a set of polynomials orthogonal with respect to the Sobolev inner product

$$\langle f, g\rangle = \int_{-1}^{3} f(x)\, g(x)\, dx + \lambda \int_{-1}^{1} f'(x)\, g'(x)\, dx + \int_{1}^{3} f'(x)\, g'(x)\, dx,$$

where $\lambda \geqslant 0$. If n is odd and λ sufficiently large, then S_n^λ has exactly one real zero. If n is even, $n \geqslant 2$, and λ sufficiently large, then S_n^λ has exactly two real zeros. This result can be generalized to a more general inner product.

1. INTRODUCTION

Recently several authors studied polynomials orthogonal with respect to a discrete Sobolev inner product of the form

$$\langle f, g\rangle = \int_a^b f(x)\, g(x)\, d\psi(x) + \lambda f'(c)\, g'(c), \tag{1.1}$$

where $\lambda \geqslant 0$, $c \in \mathbb{R}$. For a survey of the results and a complete list of references, see [5]. Let $\{S_n^\lambda\}$ denote a set of polynomials (normalized in some way) orthogonal with respect to (1.1). It has been proved that S_n^λ has at least $n-2$ different real zeros in (a, b); if $n \geqslant 3$ it is possible to choose a $c \in (a, b)$ such that S_n^λ for sufficiently large λ has 2 complex zeros [6].

Results on the zeros of polynomials orthogonal with respect to a non-discrete Sobolev inner product of the form

$$\langle f, g\rangle = \int_a^b f(x)\, g(x)\, d\psi_0(x) + \int_a^b f'(x)\, g'(x)\, d\psi_1(x)$$

are only known for special choices of ψ_0 and ψ_1 from the papers of Althammer [1], Brenner [2], and Cohen [3].

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Althammer considered the case

$$(a, b) = (-1, 1), \qquad d\psi_0(x) = dx, \qquad d\psi_1(x) = \lambda\, dx \qquad (\lambda \geqslant 0).$$

Let again $\{S_n^\lambda\}$ denotes the corresponding set of orthogonal polynomials. Althammer proved that S_n^λ has n different, real zeros in $(-1, 1)$. Cohen gave a more precise description for this situation: if $\lambda \geqslant 2/n$, then the zeros of S_n^λ interlace with the zeros of P_{n-1}, the Legendre polynomial of degree $n-1$. Brenner proved a result similar to that of Althammer for

$$(a, b) = (0, \infty), \qquad d\psi_0(x) = e^{-x}\, dx, \qquad d\psi_1(x) = \lambda e^{-x}\, dx \qquad (\lambda \geqslant 0).$$

If $\{S_n^\lambda\}$ is the set of corresponding orthogonal polynomials, then S_n^λ has n different zeros in $(0, \infty)$. Furthermore, Althammer made a remark: if

$$(a, b) = (-1, 1), \qquad d\psi_0(x) = dx, \qquad d\psi_1(x) = \begin{cases} 10dx, & -1 \leqslant x < 0, \\ dx, & 0 \leqslant x \leqslant 1, \end{cases}$$

then the polynomial of degree 2 has a zero in -1.08, i.e., outside the interval of orthogonality. Althammer did not give a result for polynomials of degree $n > 2$ in this situation. Brenner made a similar remark, also only for $n = 2$.

It is the aim of this paper to generalize the last mentioned remark of Althammer. In order to simplify the calculations we take the interval $[-1, 3]$ instead of $[-1, 1]$. In Section 2 we consider the inner product

$$\langle f, g \rangle = \int_{-1}^{3} f(x)\, g(x)\, dx + \lambda \int_{-1}^{1} f'(x)\, g'(x)\, dx + \int_{1}^{3} f'(x)\, g'(x)\, dx,$$

where $\lambda \geqslant 0$. As before let $\{S_n^\lambda\}$ denote the set of polynomials orthogonal with respect to this inner product. We prove:

if n is even, $n \geqslant 2$, and λ sufficiently large, then S_n^λ has exactly two real zeros: one in $(1, 3)$ and one in $(-3, -1)$;

if n is odd, $n \geqslant 3$, and λ sufficiently large, then S_n^λ has exactly one real zero: the zero is in $(1, 3)$.

So the structure of the zeros is quite different from the situation in the discrete case.

In Section 3 we show that the result of Section 2 can easily be generalized to an inner product of the form

$$\langle f, g \rangle = \int_{-1}^{a} f(x) g(x)\, d\psi_0(x) + \lambda \int_{-1}^{1} f'(x)\, g'(x)\, d\psi_1(x)$$
$$+ \int_{1}^{a} f'(x)\, g'(x)\, d\psi_1(x),$$

where $1 < a \leqslant \infty$, $\lambda \geqslant 0$. We prove that there exists an n_0, such that if $n \geqslant n_0$ for S_n^λ the assertion of Section 2 holds.

2. The Althammer Case

Consider the inner product

$$\langle f, g \rangle = \int_{-1}^{3} f(x)\, g(x)\, dx + \lambda \int_{-1}^{+1} f'(x)\, g'(x)\, dx + \int_{1}^{3} f'(x)\, g'(x)\, dx, \quad (2.1)$$

where $\lambda \geqslant 0$. We want to study the zeros of a set of polynomials $\{S_n^\lambda\}$ orthogonal with respect to this inner product. Since the zeros do not depend on the normalization, we may choose a normalization which simplifies the calculations. We put

$$a_{i,j} = \langle x^i, x^j \rangle, \quad (2.2a)$$

and define

$$S_0^\lambda(x) \equiv 1, \quad (2.2b)$$

$$S_n^\lambda(x) = \begin{vmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,n} \\ a_{1,0} & a_{1,1} & \cdots & a_{1,n} \\ \vdots & \vdots & & \vdots \\ a_{n-1,0} & a_{n-1,1} & \cdots & a_{n-1,n} \\ 1 & x & \cdots & x^n \end{vmatrix} \quad \text{if} \quad n \geqslant 1. \quad (2.2c)$$

Then $\{S_n^\lambda(x)\}$ is a set of polynomials orthogonal with respect to the inner product (2.1). Obviously the elements in the first row and in the first column of the determinant are independent of λ. For $i \geqslant 1$, $j \geqslant 1$, however, $a_{i,j}$ is linear in λ. Hence S_n^λ is a polynomial in λ of degree (at most) $n-1$.

Define

$$Q_n(x) = \lim_{\lambda \to \infty} \frac{1}{\lambda^{n-1}} S_n^\lambda(x) \qquad \text{for} \quad n \geqslant 2. \quad (2.3)$$

Then for large λ the zeros of S_n^λ are determined by those of Q_n. We have

$$Q_n(x) = \begin{vmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,n} \\ 0 & b_{1,1} & \cdots & b_{1,n} \\ \vdots & \vdots & & \vdots \\ 0 & b_{n-1,1} & \cdots & b_{n-1,n} \\ 1 & x & \cdots & x^n \end{vmatrix} \quad \text{if} \quad n \geqslant 2, \quad (2.4)$$

where $b_{i,j} = ij \int_{-1}^{+1} x^{i+j-2} dx$. Observe that the leading coefficient of Q_n is positive. From (2.4) it follows

$$\int_{-1}^{3} Q_n(x)\, dx = 0 \qquad (n \geqslant 2), \tag{2.5}$$

$$\int_{-1}^{+1} Q_n'(x) x^i\, dx = 0, \qquad i \in \{0, 1, ..., n-2\}, \qquad (n \geqslant 2). \tag{2.6}$$

Relation (2.6) implies

$$Q_n'(x) = \text{const. } P_{n-1}(x) \qquad \text{if} \quad n \geqslant 2.$$

Then Q_n has $n-1$ extremata in the zeros of P_{n-1} in $(-1, 1)$; outside $(-1, 1)$ Q_n is monotonic.

Define

$$R_n(x) = \int_{-1}^{x} P_{n-1}(t)\, dt \qquad \text{if} \quad n \geqslant 1, \tag{2.7}$$

then there exist $A_n > 0$ and B_n such that

$$A_n Q_n(x) = R_n(x) + B_n \qquad (n \geqslant 2). \tag{2.8}$$

By (2.5) we have

$$-4B_n = \int_{-1}^{3} R_n(x)\, dx. \tag{2.9}$$

It is well-known (see, e.g., Szegő [7, (4.7.29)]) that

$$R_n(x) = \frac{P_n(x) - P_{n-2}(x)}{2n-1} \qquad \text{if} \quad n \geqslant 2. \tag{2.10}$$

Moreover, $|P_n(x)| < 1$ if $x \in [-1, 1]$. Hence

$$|R_n(x)| \leqslant \frac{2}{2n-1} \leqslant \frac{2}{5} \qquad \text{if} \quad n \geqslant 3, \qquad x \in [-1, 1]. \tag{2.11}$$

In order to estimate $R_n(x)$ on $[1, 3]$ we remark $R_n(1) = 0$ if $n \geqslant 2$, $R_n^{(k+1)}(1) = P_{n-1}^{(k)}(1) \geqslant 0$ for all $k \geqslant 0$. By Taylor expansion around $x = 1$, we obtain

$$R_n(x) \geqslant R_n'(1)(x-1) = (x-1) \qquad \text{if} \quad n \geqslant 2, \qquad x \in [1, 3].$$

If $n \geqslant 3$, we have, using (2.9) and (2.10),

$$-4B_n = \int_1^3 R_n(x)\, dx \geqslant \int_1^3 (x-1)\, dx = 2.$$

Then

$$B_n \leqslant -\tfrac{1}{2} \qquad \text{if} \quad n \geqslant 3. \tag{2.12}$$

Now suppose $n \geqslant 3$. From (2.8), (2.11), and (2.12) it follows that

$$Q_n(x) < 0 \qquad \text{if} \quad x \in [-1, 1] \text{ and } n \geqslant 3. \tag{2.13}$$

Since $Q_n'(x) = A_n^{-1} P_{n-1}(x) > 0$ if $x \geqslant 1$, the function Q_n is strictly increasing on $[1, \infty]$. Then (2.5) and (2.13) imply that $Q_n(3) > 0$. Hence Q_n has exactly one zero in $(1, 3)$.

To complete the discussion for $n \geqslant 3$, we remark that, using $Q_n' = A_n^{-1} P_{n-1}$, Q_n is monotonic on $(-\infty, -1)$. If n is even, (2.7) and (2.8) imply that Q_n is even. Then Q_n also has a zero in $(-3, -1)$. If n is odd, Q_n has only one real zero, i.e., the zero in $(1, 3)$. Direct calculation from (2.4) gives $Q_2(x) = \text{const.}\ (x^2 - 7/3)$, so Q_2 has one zero in $(1, 3)$ and one zero in $(-3, -1)$.

We have proved the following result.

THEOREM. *Let $\{S_n^\lambda\}$ denote a set of polynomials orthogonal with respect to* (2.1). *Let $n \geqslant 2$.*

If n is odd and λ sufficiently large, then S_n^λ has exactly one real zero: the zero is in $(1, 3)$.

If n is even and λ sufficiently large, then S_n^λ has exactly two real zeros: one zero in $(1, 3)$ *and one zero in* $(-3, -1)$.

Remark. We remark that for $n \geqslant 1$ and every $\lambda \geqslant 0$, we have

$$\int_{-1}^{3} S_n^\lambda(x)\, dx = \langle 1, S_n^\lambda \rangle = 0,$$

so S_n^λ always has at least one zero in $(-1, 3)$.

3. GENERALIZATION

We generalize the results of the preceding section to the inner product

$$\langle f, g \rangle = \int_{-1}^{a} f(x)\, g(x)\, d\psi_0(x) + \lambda \int_{-1}^{+1} f'(x)\, g'(x)\, d\psi_1(x)$$

$$+ \int_{1}^{a} f'(x)\, g'(x)\, d\psi_1(x), \tag{3.1}$$

where $1 < a \leqslant \infty$, $\lambda \geqslant 0$, ψ_0 and ψ_1 are distribution functions on $[-1, a]$ such that polynomials are absolutely integrable with respect to them; we suppose that ψ_0 has at least one point of increase in $(1, a]$ and that ψ_1' exists on $(-1, 1)$ with $\psi_1'(x) \geqslant \mu > 0$ on $(-1, 1)$.

Define a set of polynomials $\{S_n^\lambda\}$ orthogonal with respect to the inner product (3.1) as before by (2.2a), (2.2b), and (2.2c). Then S_n^λ again is a polynomial in λ of degree (at most) $n-1$. Define Q_n (for $n \geqslant 2$) by (2.3); then for Q_n the determinantal representation (2.4) holds with

$$a_{0,j} = \int_{-1}^{a} x^j \, d\psi_0(x), \qquad b_{i,j} = ij \int_{-1}^{+1} x^{i+j-2} \, d\psi_1(x).$$

Again the leading coefficient of Q_n is positive. Now Q_n satisfies

$$\int_{-1}^{a} Q_n(x) \, d\psi_0(x) = 0 \qquad (n \geqslant 2), \tag{3.2}$$

$$\int_{-1}^{+1} Q_n'(x) x^i \, d\psi_1(x) = 0, \qquad i \in \{0, 1, ..., n-2\}, \qquad (n \geqslant 2). \tag{3.3}$$

Let $\{\varphi_n\}$ denote the set of polynomials orthonormal with respect to the inner product

$$(f, g) = \int_{-1}^{+1} f(x) \, g(x) \, d\psi_1(x), \tag{3.4}$$

where the leading coefficients are positive. Define

$$R_n(x) = \int_{-1}^{x} \varphi_{n-1}(t) \, dt \qquad \text{if} \quad n \geqslant 1. \tag{3.5}$$

It follows from (3.3) that there exist $A_n > 0$ and B_n such that

$$A_n Q_n(x) = R_n(x) + B_n \qquad (n \geqslant 2). \tag{3.6}$$

By (3.2), we have

$$-B_n \int_{-1}^{a} d\psi_0(x) = \int_{-1}^{a} R_n(x) \, d\psi_0(x). \tag{3.7}$$

We have, using (3.5) and the Cauchy–Schwarz inequality for $x \in [-1, 1]$,

$$|R_n(x)| \leqslant \int_{-1}^{+1} |\varphi_{n-1}(t)| \, dt \leqslant \sqrt{2} \left\{ \int_{-1}^{+1} \varphi_{n-1}^2(t) \, dt \right\}^{1/2}$$
$$\leqslant \sqrt{2} \, \mu^{-1/2} \left\{ \int_{-1}^{+1} \varphi_{n-1}^2(t) \, d\psi_1(t) \right\}^{1/2}.$$

Hence

$$|R_n(x)| \leqslant \sqrt{2}\,\mu^{-1/2} \qquad \text{if} \quad x \in [-1, 1]. \tag{3.8}$$

Since R_n is increasing on $(1, \infty)$, this implies

$$R_n(x) \geqslant -\sqrt{2}\,\mu^{-1/2} \qquad \text{if} \quad x \geqslant -1. \tag{3.9}$$

In order to estimate the right-hand side of (3.7) from below, we choose a $c \in (1, a)$ such that $\int_c^a (x-c)\,d\psi_0(x) \neq 0$. We observe $R_n^{(k+1)}(c) = \varphi_{n-1}^{(k)}(c) \geqslant 0$ for all $k \geqslant 0$. Hence, by Taylor expansion around $x = c$, we have

$$R_n(x) \geqslant R_n(c) + R_n'(c)(x-c) \qquad \text{if} \quad x \geqslant c,$$

and, using (3.9),

$$R_n(x) \geqslant -\sqrt{2}\,\mu^{-1/2} + \varphi_{n-1}(c)(x-c), \qquad x \geqslant c.$$

Then it follows from (3.7) that

$$-B_n \int_{-1}^{a} d\psi_0(x) \geqslant \int_{-1}^{c} R_n(x)\,d\psi_0(x) - \sqrt{2}\,\mu^{-1/2} \int_c^a d\psi_0(x)$$

$$+ \varphi_{n-1}(c) \int_c^a (x-c)\,d\psi_0(x),$$

and with (3.9) that

$$-B_n \int_{-1}^{a} d\psi_0(x) \geqslant -\sqrt{2}\,\mu^{-1/2} \int_{-1}^{a} d\psi_0(x) + \varphi_{n-1}(c) \int_c^a (x-c)\,d\psi_0(x). \tag{3.10}$$

Since $c > 1$, it holds (see Freud [4, p. 120]) that

$$\liminf_{n \to \infty} \sqrt[n]{\varphi_n(c)} \geqslant c + \sqrt{c^2 - 1} > 1.$$

Then the right-hand side of (3.10) exceeds $\sqrt{2}\,\mu^{-1/2} \int_{-1}^{a} d\psi_0(x)$ if n is sufficiently large. Hence

$$B_n < -\sqrt{2}\,\mu^{-1/2} \qquad \text{if } n \text{ is sufficiently large.} \tag{3.11}$$

Suppose now that n is so large that (3.11) holds. Then we can make the following observations. By (3.6), (3.8), and (3.11) we have

$$Q_n(x) < 0 \qquad \text{if} \quad x \in [-1, 1].$$

By (3.6) and (3.5), $Q_n'(x) = A_n^{-1}\varphi_{n-1}(x) > 0$ on $(1, \infty)$, hence Q_n is increasing on $(1, \infty)$. Relation (3.2) implies $Q_n(a) > 0$ and Q_n has one zero in $(1, a)$. Since $Q_n' = A_{n-1}^{-1}\varphi_{n-1}$, the function Q_n is monotonic on $(-\infty, -1)$. Then, if n is odd, Q_n has only one real zero; if n is even, Q_n has two real zeros: one in $(1, a)$ and one in $(-\infty, -1)$.

We have proved the following result.

THEOREM. *Let $\{S_n^\lambda\}$ denote a set of polynomials orthogonal with respect to* (3.1). *Then there exists an n_0, such that for $n \geqslant n_0$ the following holds.*

If n is odd and λ sufficiently large, then S_n^λ has exactly one real zero: the zero is in $(1, a)$.

If n is even and λ sufficiently large, then S_n^λ has exactly two real zeros: one zero in $(1, a)$ and one in $(-\infty, -1)$.

Remark. The n_0 in the Theorem depends on ψ_0 and ψ_1. The next example shows that it is not possible to replace n_0 by an integer independent of ψ_0 and ψ_1.

Let $d\psi_1(x) = dx$ on $[-1, a]$ as in Section 2. Then

$$R_n(x) = \text{const.}\ \frac{(x^2-1)\,P_{n-2}^{(1,1)}(x)}{2(n-1)} \qquad \text{if} \quad n \geqslant 2,$$

(compare Cohen [3, (15)]). Take a fixed $n \geqslant 2$ and define

$$V = \{x \in [-1, a],\ R_n(x) > 0\}, \qquad W = \{x \in [-1, a],\ R_n(x) < 0\}.$$

Then both V and W are not empty. Put

$$\mu_1 = \int_V R_n(x)\,dx > 0, \qquad \mu_2 = -\int_W R_n(x)\,dx > 0.$$

Finally, take a distribution function ψ_0 on $[-1, a]$ such that

$$\begin{aligned} d\psi_0(x) &= \mu_1\,dx && \text{if} \quad x \in W \\ d\psi_0(x) &= \mu_2\,dx && \text{if} \quad x \in V \\ d\psi_0(x) & && \text{is continuous on } [-1, a]. \end{aligned}$$

Then

$$\int_{-1}^{a} R_n(x)\,d\psi_1(x) = \mu_1 \int_W R_n(x)\,dx + \mu_2 \int_V R_n(x)\,dx = 0.$$

Then (3.7) implies $B_n = 0$. Hence, by (3.6), Q_n has the same zeros as R_n, i.e., Q_n has n different real zeros in $[-1, 1]$. We conclude: for every $n \geqslant 2$, there exist ψ_0 and ψ_1 such that Q_n has n different real zeros in $[-1, 1]$.

Remark. The condition $\psi_1'(x) \geqslant \mu > 0$ on $(-1, 1)$ can be weakened somewhat. Since $\varphi_{n-1}(c)$ in (3.10) is of exponential growth as $n \to \infty$, it is sufficient that there exists constants C and k such that

$$|R_n(x)| < Cn^k \qquad \text{if} \quad x \in [-1, 1].$$

This holds, for instance, if ψ_1 is a Jacobian measure (compare [7, p. 161]).

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References

1. P. Althammer, Eine Erweiterung des Orthogonalitätsbegriffes bei Polynomen und deren Anwendung auf die beste Approximation, *J. Reine Angew. Math.* **211** (1962), 192–204.
2. J. Brenner, Über eine Erweiterung des Orthogonalitätsbegriffes bei Polynomen, *in* "Constructive Theory of Functions," pp. 77–83, Akadémia Kiadó, Budapest, 1972.
3. E. A. Cohen, Zero distribution and behavior of orthogonal polynomials in the Sobolev space $W^{1,2}[-1, 1]$, *SIAM J. Math. Anal.* **6** (1975), 105–116.
4. G. Freud, "Orthogonale Polynome," Birkhäuser-Verlag, Basel, 1969.
5. F. Marcellán, M. Alfaro, and M. L. Rezola, Orthogonal polynomials on Sobolev spaces: Old and new directions, *J. Comput. Appl. Math.* **48** (1993), 113–131.
6. H. G. Meijer, On real and complex zeros of orthogonal polynomials in a discrete Sobolev space, *J. Comput. Appl. Math.* **49** (1993), 179–191.
7. G. Szegő, "Orthogonal Polynomials," 4th Ed., American Mathematical Society Colloquium Publications, Vol. 23, Amer. Math. Soc., Providence, RI, 1975. [Original version published in 1939]